

On Calderón's conjecture

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1. Introduction

This paper is a successor of [4]. In that paper we considered bilinear operators of the form

$$(1) \quad H_\alpha(f_1, f_2)(x) := \text{p.v.} \int f_1(x-t)f_2(x+\alpha t)\frac{dt}{t},$$

which are originally defined for f_1, f_2 in the Schwartz class $\mathcal{S}(\mathbb{R})$. The natural question is whether estimates of the form

$$(2) \quad \|H_\alpha(f_1, f_2)\|_p \leq C_{\alpha, p_1, p_2} \|f_1\|_{p_1} \|f_2\|_{p_2}$$

with constants C_{α, p_1, p_2} depending only on α, p_1, p_2 and $p := \frac{p_1 p_2}{p_1 + p_2}$ hold. The first result of this type is proved in [4], and the purpose of the current paper is to extend the range of exponents p_1 and p_2 for which (2) is known. In particular, the case $p_1 = 2, p_2 = \infty$ is solved to the affirmative. This was originally considered to be the most natural case and is known as Calderón's conjecture [3].

We prove the following theorem:

THEOREM 1. *Let $\alpha \in \mathbb{R} \setminus \{0, -1\}$ and*

$$(3) \quad 1 < p_1, p_2 \leq \infty,$$

$$(4) \quad \frac{2}{3} < p := \frac{p_1 p_2}{p_1 + p_2} < \infty.$$

Then there is a constant C_{α, p_1, p_2} such that estimate (2) holds for all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$.

If $\alpha = 0, -1, \infty$, then we obtain the bilinear operators

$$H(f_1) \cdot f_2, H(f_1 \cdot f_2), f_1 \cdot H(f_2),$$

*The first author acknowledges support from the NSF. Both authors have been supported by NATO travel grants and the Volkswagen Stiftung (RiP program in Oberwolfach).

the last one by replacing t with t/α and taking a weak limit as α tends to infinity. Here H is the ordinary linear Hilbert transform, and \cdot is pointwise multiplication. The L^p -bounds of these operators are easy to determine and quite different from those in the theorem. This suggests that the behaviour of the constant C_{α,p_1,p_2} is subtle near the exceptional values of α . It would be of interest to know that the constant is independent of α for some choices of p_1 and p_2 .

We do not know that the condition $\frac{2}{3} < p$ is necessary in the theorem. But it is necessary for our proof. An easy counterexample shows that the unconditionality in inequality (6) already requires $\frac{2}{3} \leq p$. The cases of (p_1, p_2) being equal to $(1, \infty)$, $(\infty, 1)$, or (∞, ∞) have to be excluded from the theorem, since the ordinary Hilbert transform is not bounded on L^1 or L^∞ .

We assume the reader as somewhat familiar with the results and techniques of [4]. The differences between the current paper and [4] manifest themselves in the overall organization and the extension of the counting function estimates to functions in L^q with $q < 2$.

The authors would like to thank the referee for various corrections and suggestions towards improving this exposition.

2. Preliminary remarks on the exponents

Call a pair (p_1, p_2) good, if for all $\alpha \in \mathbb{R} \setminus \{0, -1\}$ there is a constant C_{α,p_1,p_2} such that estimate (2) holds for all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$. In this section we discuss interpolation and duality arguments. These, together with the known results from [4], show that instead of Theorem 1 it suffices to prove:

PROPOSITION 1. *If $1 < p_1, p_2 < 2$ and $\frac{2}{3} < \frac{p_1 p_2}{p_1 + p_2}$, then (p_1, p_2) is good.*

In [4] the following is proved:

PROPOSITION 2. *If $2 < p_1, p_2 < \infty$ and $1 < \frac{p_1 p_2}{p_1 + p_2} < 2$, then (p_1, p_2) is good.*

Strictly speaking, this proposition is proved in [4] only in the case $\alpha = 1$, but this restriction is inessential. The necessary modifications to obtain the full result appear in the current paper in Section 3. Therefore we take Proposition 2 for granted.

The next lemma follows by complex interpolation as in [1]. The authors are grateful to E. Stein for pointing out this reference to them.

LEMMA 1. *Let $1 < p_1, p_2, q_1, q_2 \leq \infty$ and assume that (p_1, p_2) and (q_1, q_2) are good. Then*

$$\left(\frac{\theta}{p_1} + \frac{1-\theta}{q_1}, \frac{\theta}{p_2} + \frac{1-\theta}{q_2} \right)$$

is good for all $0 < \theta < 1$.

Next we need a duality lemma.

LEMMA 2. *Let $1 < p_1, p_2 < \infty$ such that $\frac{p_1 p_2}{p_1 + p_2} \geq 1$. If (p_1, p_2) is good, then so are the pairs*

$$\left(p_1, \left(\frac{p_1 p_2}{p_1 + p_2}\right)'\right) \quad \text{and} \quad \left(\left(\frac{p_1 p_2}{p_1 + p_2}\right)', p_2\right).$$

Here p' denotes as usual the dual exponent of p . To prove the lemma, fix $\alpha \in \mathbb{R} \setminus \{0, -1\}$ and $f_1 \in \mathcal{S}(\mathbb{R})$ and consider the linear operator $H_\alpha(f_1, -)$. The formal adjoint of this operator with respect to the natural bilinear pairing is

$$\operatorname{sgn}(1 + \alpha) H_{-\frac{\alpha}{1+\alpha}}(f_1, -),$$

as the following lines show:

$$\begin{aligned} & \int \left(\text{p.v.} \int f_1(x-t) f_2(x+\alpha t) \frac{1}{t} dt \right) f_3(x) dx \\ &= \text{p.v.} \int \int f_1(x-\alpha t-t) f_2(x) f_3(x-\alpha t) dx \frac{1}{t} dt \\ &= \operatorname{sgn}(1 + \alpha) \int \left(\text{p.v.} \int f_1(x-t) f_3\left(x - \frac{\alpha}{1+\alpha} t\right) \frac{1}{t} dt \right) f_2(x) dx. \end{aligned}$$

Similarly, we observe that for fixed f_2 the formal adjoint of $H_\alpha(-, f_2)$ is $-H_{-1-\alpha}(-, f_2)$. This proves Lemma 2 by duality.

Now we are ready to prove estimate (2) in the remaining cases, i.e., for those pairs (p_1, p_2) for which one of p_1, p_2 is smaller or equal two, and the other one is greater or equal two. In this case the constraint on p is automatically satisfied. By symmetry it suffices to do this for $p_1 \in]1, 2]$ and $p_2 \in [2, \infty]$. First observe that the pairs $(3, 3)$ and $(3/2, 3/2)$ are good by the above propositions. Then the pairs $(2, 2)$ and $(2, \infty)$ are good by interpolation and duality. Let P be the set of all $p_1 \in]1, 2]$ such that the pair (p_1, p_2) is good for all $p_2 \in [2, \infty]$. The previous observations show that $2 \in P$. Define $p := \inf P$ and assume $p > 1$. Pick a small $\varepsilon > 0$ and a $p_1 \in P$ with $p_1 < p + \varepsilon$. If ε is small enough, we can interpolate the good pairs (p_1, ε^{-1}) and $(1 + \varepsilon, 2 - \varepsilon)$ to obtain a good pair of the form $(q_\varepsilon, q'_\varepsilon)$. Since $\lim_{\varepsilon \rightarrow 0} q_\varepsilon = \frac{3p-2}{2p-1} < p$ we have $q_\varepsilon < p$ provided ε is small enough. By duality we see that the pair (q, ∞) is good, and by Proposition 1 there is a $p_2 < 2$ such that (q, p_2) is good. By interpolation $q \in P$ follows. This is a contradiction to $p = \inf P$; therefore the assumption $p > 1$ is false and we have $\inf P = 1$. Again by interpolation we observe $P =]1, 2]$, which finishes the prove of estimate (2) for the remaining exponents.

3. Time-frequency decomposition of H_α

In this section we write the bilinear operators H_α approximately as finite sums over rank one operators, each rank one operator being well localized in time and frequency. We mostly follow the corresponding section in [4], adopting the basic notation and definitions from there such as that of a phase plane representation.

In contrast to [4] we work out how the decomposition and the constants depend on α , and we add an additional assumption (iv) in Proposition 3 which is necessary to prove L^p - estimates for $p < 2$. The reader should think of the functions $\theta_{\xi,i}$ in this assumption as being exponentials $\theta_{\xi,i}(x) = e^{i\eta_i x}$ for certain frequencies $\eta_i = \eta_i(\xi)$.

PROPOSITION 3. *Assume we are given exponents $1 < p_1, p_2 < 2$ such that $\frac{p_1 p_2}{p_1 + p_2} > \frac{2}{3}$, and we are given a constant C_m for each integer $m \geq 0$. Then there is a constant C depending on these data such that the following holds:*

Let S be a finite set, $\phi_1, \phi_2, \phi_3 : S \rightarrow \mathcal{S}(\mathbb{R})$ be injective maps, and $I, \omega_1, \omega_2, \omega_3 : S \mapsto \mathcal{J}$ be maps such that $I(S)$ is a grid, $\mathcal{J}_\omega := \omega_1(S) \cup \omega_2(S) \cup \omega_3(S)$ is a grid, and the following properties (i)–(iv) hold for all $i \in \{1, 2, 3\}$:

(i) *The map*

$$\rho_i : \phi_i(S) \rightarrow \mathcal{R}, \phi_i(s) \mapsto I(s) \times \omega_i(s)$$

is a phase plane representation with constants C_m .

(ii) $\omega_i(s) \cap \omega_j(s) = \emptyset$ for all $s \in S$ and $j \in \{1, 2, 3\}$ with $i \neq j$.

(iii) If $\omega_i(s) \subset J$ and $\omega_i(s) \neq J$ for some $s \in S$, $J \in \mathcal{J}_\omega$, then $\omega_j(s) \subset J$ for all $j \in \{1, 2, 3\}$.

(iv) To each $\xi \in \mathbb{R}$ there is associated a measurable function $\theta_{\xi,i} : \mathbb{R} \rightarrow \{z \in \mathbb{C} : |z| = 1\}$ such that for all $s \in S$, $j \in \{1, 2, 3\}$ and $J \in I(S)$ the following holds: If $\xi \in \omega_j(s)$, $|J| \leq |I(s)|$, then

$$(5) \quad \inf_{\lambda \in \mathbb{C}} \|\phi_i(s) - \lambda \theta_{\xi,i}\|_{L^\infty(J)} \leq C_0 |J| |I(s)|^{-\frac{3}{2}} \left(1 + \frac{|c(J) - c(I(s))|}{|I(s)|} \right)^{-2}.$$

For all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$ and all maps $\varepsilon : S \rightarrow [-1, 1]$, we then have:

$$(6) \quad \left\| \sum_{s \in S} \varepsilon(s) |I(s)|^{-\frac{1}{2}} \langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \phi_3(s) \right\|_{\frac{p_1 p_2}{p_1 + p_2}} \leq C \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

In the rest of this section we prove Proposition 2 under the assumption that Proposition 3 above is true. Let $1 < p_1, p_2 < 2$ with $\frac{2}{3} < p := \frac{p_1 p_2}{p_1 + p_2}$ and $\alpha \in \mathbb{R} \setminus \{0, -1\}$.

Let L be the smallest integer larger than

$$2^{10} \max \left\{ |\alpha|, \frac{1}{|\alpha|}, \frac{1}{|1+\alpha|} \right\}.$$

The dependence on α will enter into our estimate via a polynomial dependence on L .

Define $\varepsilon := L^{-3}$. Pick a function $\psi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\psi}$ is supported in $[L^3 - 1, L^3 + 1]$ and

$$\sum_{k \in \mathbb{Z}} \hat{\psi}(2^{\varepsilon k} \xi) = 1 \quad \text{for all } \xi > 0.$$

Define

$$\psi_k(x) := 2^{-\frac{\varepsilon k}{2}} \psi(2^{-\varepsilon k} x)$$

and

$$(7) \quad \tilde{H}_\alpha(f_1, f_2)(x) := \sum_{k \in \mathbb{Z}} 2^{-\frac{\varepsilon k}{2}} \int_{\mathbb{R}} f_1(x-t) f_2(x+\alpha t) \psi_k(t) dt.$$

It suffices to prove boundedness of \tilde{H}_α . Pick a $\varphi \in \mathcal{S}(\mathbb{R})$ such that $\hat{\varphi}$ is supported in $[-1, 1]$ and

$$(8) \quad \sum_{n, l \in \mathbb{Z}} \left\langle f, \varphi_{k, n, \frac{l}{2}} \right\rangle \varphi_{k, n, \frac{l}{2}} = f$$

for all Schwartz functions f , where

$$\varphi_{\kappa, n, l}(x) := 2^{-\frac{\varepsilon \kappa}{2}} \varphi(2^{-\varepsilon \kappa} x - n) e^{2\pi i 2^{-\varepsilon \kappa} x l}.$$

We apply this formula three times in (7) to obtain:

$$(9) \quad \tilde{H}_\alpha(f_1, f_2)(x) = \sum_{k, n_1, n_2, n_3, l_1, l_2, l_3 \in \mathbb{Z}} C_{k, n_1, n_2, n_3, l_1, l_2, l_3} H_{k, n_1, n_2, n_3, l_1, l_2, l_3}(f_1, f_2)(x)$$

with

$$H_{k, n_1, n_2, n_3, l_1, l_2, l_3}(f_1, f_2)(x) := 2^{-\frac{\varepsilon k}{2}} \left\langle f_1, \varphi_{k, n_1, \frac{l_1}{2}} \right\rangle \left\langle f_2, \varphi_{k, n_2, \frac{l_2}{2}} \right\rangle \varphi_{k, n_3, \frac{l_3}{2}}(x)$$

and

$$(10) \quad C_{k, n_1, n_2, n_3, l_1, l_2, l_3} := \int \int \varphi_{k, n_1, \frac{l_1}{2}}(x-t) \varphi_{k, n_2, \frac{l_2}{2}}(x+\alpha t) \varphi_{k, n_3, \frac{l_3}{2}}(x) \psi_k(t) dt dx.$$

The proof of the following lemma is a straightforward calculation as in [4].

LEMMA 3. *There is a constant C depending on ϕ and ψ such that*

$$(11) \quad |C_{k, n_1, n_2, n_3, l_1, l_2, l_3}| \leq C \left(1 + \frac{1}{L} \text{diam}\{n_1, n_2, n_3\} \right)^{-100}.$$

Moreover,

$$C_{k,n_1,n_2,n_3,l_1,l_2,l_3} = 0,$$

unless

$$(12) \quad l_1 \in \left[\left(-\frac{\alpha}{1+\alpha} l_3 + \frac{2}{1+\alpha} L^3 \right) - L, \left(-\frac{\alpha}{1+\alpha} l_3 + \frac{2}{1+\alpha} L^3 \right) + L \right]$$

and

$$(13) \quad l_2 \in \left[\left(-\frac{1}{1+\alpha} l_3 - \frac{2}{1+\alpha} L^3 \right) - L, \left(-\frac{1}{1+\alpha} l_3 - \frac{2}{1+\alpha} L^3 \right) + L \right].$$

Now we can reduce Proposition 2 to the following lemma:

LEMMA 4. *There is a constant C depending on p_1 , p_2 , φ , and ψ such that the following holds:*

Let $\nu > 0$ be an integer and let S be a finite subset of \mathbb{Z}^3 such that for $(k, n, l), (k', n', l') \in S$ the following three properties are satisfied:

$$(14) \quad \text{If } k \neq k' \text{ ,} \quad \text{then } |k - k'| > L^{10},$$

$$(15) \quad \text{if } n \neq n' \text{ ,} \quad \text{then } |n - n'| > L^{10}\nu,$$

$$(16) \quad \text{if } l \neq l' \text{ ,} \quad \text{then } |l - l'| > L^{10}.$$

Let ν_1, ν_2 be integers with $1 + \max\{|\nu_1|, |\nu_2|\} = \nu$ and let $\lambda_1, \lambda_2 : \mathbb{Z} \rightarrow \mathbb{Z}$ be functions such that $l_1 := \lambda_1(l_3)$ satisfies (12) and $l_2 := \lambda_2(l_3)$ satisfies (13) for all $l_3 \in \mathbb{Z}$. Then we have for all $f_1, f_2 \in \mathcal{S}(\mathbb{R})$ and all maps $\varepsilon : S \rightarrow [-1, 1]$:

$$(17) \quad \left\| \sum_{(k,n,l) \in S} \varepsilon(k, n, l) H_{k, n+\nu_1, n+\nu_2, n, \lambda_1(l), \lambda_2(l), l}(f_1, f_2) \right\|_p \leq CL^{30} \nu^{10} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Before proving the lemma we show how it implies boundedness of \tilde{H}_α and therefore proves Proposition 2. First observe that the lemma also holds without the finiteness condition on S . We can also remove conditions (14), (15), and (16) on S at the cost of some additional powers of L and ν , so that the conclusion of the lemma without these hypotheses is

$$(18) \quad \left\| \sum_{(k,n,l) \in S} \varepsilon(k, n, l) H_{k, n+\nu_1, n+\nu_2, n, \lambda_1(l), \lambda_2(l), l}(f_1, f_2) \right\|_p \leq CL^{100} \nu^{20} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Here we have used the quasi triangle inequality for L^p which is uniform for $p > \frac{2}{3}$.

Observe that (18) and (11) imply

$$(19) \quad \left\| \sum_{(k,n,l) \in S} C_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l} H_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}(f_1, f_2) \right\|_p \\ \leq CL^{200} \nu^{-50} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Conditions (12) and (13) give a bound on the number of values the functions λ_1 and λ_2 can take at a fixed l_3 so that the coefficient $C_{k,n+\nu_1,n+\nu_2,n,\lambda_1(l),\lambda_2(l),l}$ does not vanish. Moreover there are of the order ν pairs ν_1, ν_2 such that $1 + \max\{|\nu_1|, |\nu_2|\} = \nu$. Hence,

$$\left\| \sum_{(k,n,l) \in S, n_1, n_2, l_1, l_2 \in \mathbb{Z}, 1 + \max\{|n-n_1|, |n-n_2|\} = \nu} C_{k,n_1,n_2,n,l_1,l_2,l} H_{k,n_1,n_2,n,l_1,l_2,l}(f_1, f_2) \right\|_p \\ \leq CL^{300} \nu^{-20} \|f_1\|_{p_1} \|f_2\|_{p_2}.$$

Summing over all ν gives boundedness of \tilde{H}_α .

It remains to prove Lemma 4. Clearly we intend to do this by applying Proposition 3. Fix data $S, \nu, \nu_1, \nu_2, \lambda_1, \lambda_2$ as in Lemma 4. Define functions $\phi_i : S \mapsto \mathcal{S}(\mathbb{R})$ as follows:

$$\begin{aligned} \phi_1(k, n, l) &:= L^{-10} \nu^{-2} \varphi_{k, n+\nu_1, \frac{\lambda_1(l)}{2}}, \\ \phi_2(k, n, l) &:= L^{-10} \nu^{-2} \varphi_{k, n+\nu_2, \frac{\lambda_2(l)}{2}}, \\ \phi_3(k, n, l) &:= L^{-10} \nu^{-2} \varphi_{k, n, \frac{l}{2}}. \end{aligned}$$

If E is a subset of \mathbb{R} and $x \neq 0$ a real number we use the notation $x \cdot E := \{xy \in \mathbb{R} : y \in E\}$. This is not to be confused with the previously defined xI for positive x and intervals I . Pick three maps $\omega_1, \omega_2, \omega_3 : S \rightarrow \mathcal{J}$ such that the following properties (20)–(25) are satisfied for all $s = (k, n, l) \in S$:

$$(20) \quad -\frac{1+\alpha}{\alpha} \cdot \text{supp}(\widehat{\phi_1(s)}) \subset \omega_1(s),$$

$$(21) \quad -(1+\alpha) \cdot \text{supp}(\widehat{\phi_2(s)}) \subset \omega_2(s),$$

$$(22) \quad \text{supp}(\widehat{\phi_3(s)}) \subset \omega_3(s),$$

$$(23) \quad 2^{-\varepsilon(k+1)} L \leq |\omega_i(s)| \leq 2^{-\varepsilon k} L \text{ for } i = 1, 2, 3,$$

$$(24) \quad \mathcal{J}_\omega := \omega_1(S) \cup \omega_2(S) \cup \omega_3(S) \text{ is a grid,}$$

and, for all $i, j \in \{1, 2, 3\}$,

(25) If $\omega_i(s) \subset J$ and $\omega_i(s) \neq J$ for some $J \in \mathcal{J}_\omega$, then $\omega_j(s) \subset J$.

The existence of such a triple of maps is proved as in [4].

Next pick a map $I : S \rightarrow \mathcal{J}$ which satisfies the following three properties (26)–(28) for all $s = (k, n, l) \in S$:

$$(26) \quad |c(I(s)) - 2^{\varepsilon k} n| \leq 2^{\varepsilon k} \nu,$$

$$(27) \quad 2^4 2^{\varepsilon k} \nu \leq |I(s)| \leq 2^\varepsilon 2^4 2^{\varepsilon k} \nu,$$

$$(28) \quad I(S) \text{ is a grid.}$$

The existence of such a map is again proved as in [4].

Now Lemma 4 follows immediately from the fact that the data S , ϕ_1 , ϕ_2 , ϕ_3 , I , ω_1 , ω_2 , and ω_3 satisfy the hypotheses of Proposition 3. The verification of these hypotheses is as in [4] except for hypothesis (iv).

We prove hypothesis (iv) for $i = 1$, the other cases being similar. Define for $\xi \in \mathbb{R}$:

$$\theta_{\xi,1}(x) := e^{-2\pi i \frac{\alpha}{\alpha+1} x \xi}.$$

Pick $s = (k, n, l) \in S$. Obviously,

$$\nu^{-2} \varphi_{k,n+\nu_1,0}(x) \leq C |I(s)|^{-\frac{1}{2}} \left(1 + \frac{|x - c(I(s))|}{|I(s)|} \right)^{-2}$$

and

$$\nu^{-2} (\varphi_{k,n+\nu_1,0})'(x) \leq C |I(s)|^{-\frac{3}{2}} \left(1 + \frac{|x - c(I(s))|}{|I(s)|} \right)^{-2}.$$

Now let $\xi \in \omega_j(s)$. By choice of $\theta_{\xi,1}$ we see that the function

$$\varphi_{k,n+\nu_1,\frac{\lambda_1(l)}{2}} \theta_{\xi,1}^{-1}$$

arises from $\varphi_{k,n+\nu_1,0}$ by modulating with a frequency which is contained in $L^{10}[-|I(s)|^{-1}, |I(s)|^{-1}]$. Therefore,

$$(\phi_1(s) \theta_{\xi,1}^{-1})'(x) \leq C |I(s)|^{-\frac{3}{2}} \left(1 + \frac{|x - c(I(s))|}{|I(s)|} \right)^{-2}.$$

Now let $J \in I(S)$ with $|J| \leq |I(s)|$. Then we have

$$\begin{aligned} \inf_{\lambda} \|\phi_1(s) \theta_{\xi,1}^{-1} - \lambda\|_{L^\infty(J)} &\leq |J| \left\| (\phi_1(s) \theta_{\xi,1}^{-1})' \right\|_{L^\infty(J)} \\ &\leq C |J| |I(s)|^{-\frac{3}{2}} \left(1 + \frac{|c(J) - c(I(s))|}{|I(s)|} \right)^{-2}. \end{aligned}$$

This proves hypothesis (iv), and therefore finishes the reduction of Proposition 2 to Proposition 3.

4. Reduction to a symmetric statement

The following proposition is a variant of Proposition 3 which is symmetric in the indices 1, 2, and 3.

PROPOSITION 4. *Let $1 < p_1, p_2, p_3 < 2$ be exponents with*

$$1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 2$$

and let $C_m > 0$ for $m \geq 0$. Then there are constants $C, \lambda_0 > 0$ such that the following holds: Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2, \omega_3$ be as in Proposition 3, let $f_i, i = 1, 2, 3$ be Schwartz functions with $\|f_i\|_{p_i} = 1$, and define

$$E := \left\{ x \in \mathbb{R} : \max_i (M_{p_i}(Mf_i)(x)) \geq \lambda_0 \right\}.$$

Then we have

$$\sum_{s \in S: I(s) \not\subset E} |I(s)|^{-\frac{1}{2}} |\langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \langle f_3, \phi_3(s) \rangle| \leq C.$$

We now prove that Proposition 3 follows from Proposition 4.

Let $1 < p_1, p_2 < 2$ and assume

$$p := \frac{p_1 p_2}{p_1 + p_2} > \frac{2}{3}.$$

Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2, \omega_3, \varepsilon$ be as in the proposition and define for each $S' \subset S$

$$H_{S'}(f_1, f_2) = \sum_{s \in S'} \varepsilon(s) |I|^{-\frac{1}{2}} \langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle \phi_3(s).$$

By Marcinkiewicz interpolation ([2]), it suffices to prove a corresponding weak-type estimate instead of (6). By linearity and scaling invariance it suffices to prove that there is a constant C such that for $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$ we have

$$|\{x \in \mathbb{R} : |H_S(f_1, f_2)(x)| \geq 2\}| \leq C.$$

Pick an exponent p_3 such that the triple p_1, p_2, p_3 satisfies the conditions of Proposition 4, and let λ_0 be as in this proposition. Let f_1 and f_2 be Schwartz functions with $\|f_1\|_{p_1} = \|f_2\|_{p_2} = 1$.

Define

$$E_0 := \{x : \max\{M_{p_1}(Mf_1)(x), M_{p_2}(Mf_2)(x)\} \geq \lambda_0\}.$$

and

$$\begin{aligned} E_{\text{in}} &:= \left\{ x \in \mathbb{R} : \left| H_{\{s \in S: I(s) \subset E_0\}}(f_1, f_2)(x) \right| \geq 1 \right\}, \\ E_{\text{out}} &:= \left\{ x \in \mathbb{R} : \left| H_{\{s \in S: I(s) \not\subset E_0\}}(f_1, f_2)(x) \right| \geq 1 \right\}. \end{aligned}$$

It suffices to bound the measures of E_{in} and E_{out} by constants. We first estimate that of E_{out} using Proposition 4. Let $\delta > 0$ be a small number and let $\theta : [0, \infty) \rightarrow [0, 1]$ be a smooth function which vanishes on the interval $[0, 1 - \delta]$ and is constant equal to 1 on $[1, \infty)$. Extend this function to the complex plane by defining in polar coordinates $\theta(re^{i\phi}) := \theta(r)e^{-i\phi}$. Assume that δ is chosen sufficiently small to give

$$|E_{\text{out}}|^{\frac{1}{p_3}} \leq \left\| \theta \left(H_{\{s \in S : I(s) \not\subset E_0\}}(f_1, f_2) \right) \right\|_{p_3} \leq 2|E_{\text{out}}|^{\frac{1}{p_3}}.$$

Define

$$f_3 := \frac{\theta \left(H_{\{s \in S : I(s) \not\subset E_0\}}(f_1, f_2) \right)}{\left\| \theta \left(H_{\{s \in S : I(s) \not\subset E_0\}}(f_1, f_2) \right) \right\|_{p_3}}.$$

We can assume that $|E_{\text{out}}| > \lambda_0^{-p_3}$, because otherwise nothing is to prove. This assumption implies $\|M_{p_3}(Mf_3)\|_\infty < \lambda_0$. By applying Proposition 4, we obtain:

$$|E_{\text{out}}|^{1-\frac{1}{p_3}} \leq 2 \left| \int H_{\{s \in S : I(s) \not\subset E_0\}}(f_1, f_2)(x) f_3(x) dx \right| \leq C.$$

Therefore $|E_{\text{out}}|$ is bounded by a constant.

It remains to estimate the measure of the set E_{in} , which is an elementary calculation. We need the following lemma:

LEMMA 5. *Let J be an interval and define*

$$S_J := \{s \in S : I(s) = J\}.$$

Then for all $m > 0$ there is a C_m such that for all $A > 1$ and $f_1, f_2 \in \mathcal{S}(\mathbb{R})$ we have:

$$\|H_{S_J}(f_1, f_2)\|_{L^1((AJ)^c)} \leq C_m |J| A^{-m} \left(\inf_{x \in J} M_{p_1} f_1(x) \right) \left(\inf_{x \in J} M_{p_2} f_2(x) \right).$$

We prove the lemma for $|J| = 1$, which suffices by homogeneity. For $m \geq 0$ define the weight

$$w_m(x) := (1 + \text{dist}(x, J))^m.$$

Then for $1 \leq r < 2$ we obtain the estimates

$$(29) \quad \left\| \sum_{s \in S_J} \alpha_s \phi_i(s) \right\|_{L^{r'}(\omega_m)} \leq C_m \left\| (\alpha_s)_{s \in S_J} \right\|_{l^r(S_J)}$$

and

$$(30) \quad \left\| (\langle f, \phi_i(s) \rangle)_{s \in S_J} \right\|_{l^{r'}(S_J)} \leq C_m \|f\|_{L^r(\omega_m^{-1})},$$

which follow easily by interpolation ([6]) from the trivial weighted estimate at $r = 1$ and the nonweighted estimate at $r = 2$.

Now define r by

$$\frac{1}{r} = \frac{1}{p_1'} + \frac{1}{p_2'};$$

in particular we have $1 < r < 2$. By writing $H_{S_J}(f_1, f_2) = (H_{S_J}(f_1, f_2)w_m^{\frac{1}{r'}})w_m^{-\frac{1}{r'}}$ and applying Hölder we have for large m :

$$\|H_{S_J}(f_1, f_2)\|_{L^1((AJ)^c)} \leq C_M A^{-M} \|H_{S_J}(f_1, f_2)\|_{L^{r'}(w_m)}.$$

Here M depends on m and r and can be made arbitrarily large by picking m accordingly. By estimates (29) and (30) we can estimate the previously displayed expression further by

$$\begin{aligned} &\leq C_M A^{-M} \left\| (\langle f_1, \phi_1(s) \rangle \langle f_2, \phi_2(s) \rangle)_{s \in S_J} \right\|_{l^r(S_J)} \\ &\leq C_M A^{-M} \left\| (\langle f_1, \phi_1(s) \rangle)_{s \in S_J} \right\|_{l^{p_1'}(S_J)} \left\| (\langle f_2, \phi_2(s) \rangle)_{s \in S_J} \right\|_{l^{p_2'}(S_J)} \\ &\leq C_M A^{-M} \|f_1\|_{L^{p_1}(w_{10}^{-1})} \|f_2\|_{L^{p_2}(w_{10}^{-1})} \\ &\leq C_M A^{-M} \left(\inf_{x \in J} M_{p_1} f_1(x) \right) \left(\inf_{x \in J} M_{p_2} f_2(x) \right). \end{aligned}$$

This finishes the proof of Lemma 5.

We return to the estimate of the set E_{in} . Define

$$E' := E_0 \cup \bigcup_{J \in I(S): J \subset E} 4J.$$

Since $|E'| \leq 5|E_0| \leq C$, it suffices to prove

$$(31) \quad \|H_{\{s \in S: I(s) \subset E_0\}}(f_1, f_2)\|_{L^1(E'^c)} \leq C.$$

Fix $k > 1$ and define

$$\mathcal{I}_k := \{J \in I(S) : J \subset E_0, 2^k J \subset E', 2^{k+1} J \not\subset E'\}.$$

Let $J \in \mathcal{I}_k$. Then for $i = 1, 2$ we have:

$$\inf_{x \in J} M_{p_i} f_i(x) \leq 2^{k+1} \inf_{x \in 2^{k+1} J} M_{p_i} f_i(x) \leq 2^{k+1},$$

since outside the set E' the maximal function is bounded by 1. Hence, by the previous lemma,

$$\|H_{S_J}(f_1, f_2)\|_{L^1((E')^c)} \leq C_m |J| 2^{-km}.$$

Since $I(S)$ is a grid, it is easy to see that the intervals in \mathcal{I}_k are pairwise disjoint; hence we have

$$\left\| H_{\{s \in S, I(s) \in \mathcal{I}_k\}}(f_1, f_2) \right\|_{L^1((E')^c)} \leq C_m |E_0| 2^{-km}.$$

By summing over all $k > 1$ we prove (31). This finishes the estimate of the set $|E_{\text{in}}|$ and therefore the reduction of Proposition 3 to Proposition 4.

5. The combinatorics on the set S

We prove Proposition 4. Let $1 < p_1, p_2, p_3 < 2$ be exponents with

$$1 < \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} < 2.$$

Let $\eta > 0$ be the largest number such that $\frac{1}{\eta}$ is an integer and

$$\eta \leq 2^{-100} \left(2 - \sum_i \frac{1}{p_i} \right) \min_j \left(1 - \frac{1}{p_j} \right).$$

Let $S, \phi_1, \phi_2, \phi_3, I, \omega_1, \omega_2$, and ω_3 be as in Propositions 3 and 4. Let f_i , $i = 1, 2, 3$ be Schwartz functions with $\|f_i\|_{p_i} = 1$. Without loss of generality we can assume that for all $s \in S$,

$$(32) \quad I(s) \not\subset \left\{ x \in \mathbb{R} : \max_i (M_{p_i}(Mf_i)(x)) \geq \lambda_0 \right\},$$

where λ_0 is a constant which we will specify later.

Define a partial order \ll on the set of rectangles by

$$(33) \quad J_1 \times J_2 \ll J'_1 \times J'_2, \text{ if } J_1 \subset J'_1 \text{ and } J'_2 \subset J_2.$$

A subset $T \subset S$ is called a *tree of type i* , if the set $\rho_i(T)$ has exactly one maximal element with respect to \ll . This maximal element is called the *base* of the tree T and is denoted by s_T . Define $J_T := I(s_T)$.

Define $S_{-1} := S$. Let $k \geq 0$ be an integer and assume by recursion that we have already defined S_{k-1} . Define

$$S_k := S_{k-1} \setminus \bigcup_{i,j=1}^3 \left(\bigcup_{l=0}^{\infty} T_{k,i,j,l} \right),$$

where the sets $T_{k,i,j,l}$ are defined as follows. Let $k \geq 0$ and $i, j \in \{1, 2, 3\}$ be fixed. Let $l \geq 0$ be an integer and assume by recursion that we have already defined $T_{k,i,j,\lambda}$ for all integers λ with $0 \leq \lambda < l$. If one of the sets $T_{k,i,j,\lambda}$ with $\lambda < l$ is empty, then define $T_{k,i,j,l} := \emptyset$. Otherwise let \mathcal{F} denote the set of all trees T of type i which satisfy the following conditions (34)–(36):

$$(34) \quad T \subset S_{k-1} \setminus \bigcup_{\lambda < l} T_{k,i,j,\lambda},$$

$$(35) \quad \text{if } i = j, \text{ then } |\langle f_j, \phi_j(s) \rangle| \geq 2^{-\eta k} 2^{-\frac{k}{p_j}} |I(s)|^{\frac{1}{2}} \text{ for all } s \in T,$$

$$(36) \quad \text{if } i \neq j, \text{ then } \left\| \left(\sum_{s \in T} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 \geq 2^4 2^{-\frac{k}{p_j}} |J_T|.$$

If \mathcal{F} is empty, then we define $T_{k,i,j,l} := \emptyset$. Otherwise define \mathcal{F}_{\max} to be the set of all $T_{\max} \in \mathcal{F}$ which satisfy:

$$(37) \quad \text{if } T \in \mathcal{F}, \text{ } T_{\max} \subset T, \text{ then } T = T_{\max}.$$

Choose $T_{k,i,j,l} \in \mathcal{F}_{\max}$ such that for all $T \in \mathcal{F}_{\max}$,

$$(38) \quad \text{if } i < j, \quad \text{then } \omega_i(s_{T_{k,i,j,l}}) \not\prec \omega_i(s_T),$$

$$(39) \quad \text{if } i > j, \quad \text{then } \omega_i(s_T) \not\prec \omega(s_{T_{k,i,j,l}}).$$

Here $[a, b] \not\prec [a', b']$ means $b > a'$. Observe that $T_{k,i,j,l}$ actually satisfies (38) and (39) for all $T \in \mathcal{F}$. This finishes the definition of the sets $T_{k,i,j,l}$ and S_k .

Since S is finite, $T_{k,i,j,l} = \emptyset$ for sufficiently large l . In particular, each $s \in S_k$ satisfies

$$(40) \quad |\langle f_i, \phi_i(s) \rangle| \leq 2^{-\eta k} 2^{-\frac{k}{p_i'}} |I(s)|^{\frac{1}{2}}$$

for all i , since the set $\{s\}$ is a tree of type i which by construction of S_k does not satisfy (35) for $j = i$. Similarly for $j \neq i$ each tree $T \subset S_k$ of type i satisfies

$$(41) \quad \left\| \left(\sum_{s \in T} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 \leq 2^4 2^{-\frac{k}{p_j'}} |J_T|.$$

Moreover, (40) implies that the intersection of all S_k contains only elements s with $\prod_j \langle f_j, \phi_j(s) \rangle = 0$.

Let $k \leq \eta^{-2}$ and assume $T_{k,i,j,l}$ is a tree. Observe that (35) and (36) together with Lemma 6 in Section 7 provide a lower bound on the maximal function $M_{p_j}(Mf_j)(x)$ for $x \in J_{T_{k,i,j,l}}$. This lower bound depends only on η , p_j and the constants C_m of the phase plane representation. Therefore if we choose the constant λ_0 in (32) small enough depending on η , p_j , and C_m , it then is clear that $T_{k,i,j,l} = \emptyset$ for $k \leq \eta^{-2}$.

Now we have

$$\begin{aligned} \sum_{s \in S} |I(s)|^{-\frac{1}{2}} \prod_j |\langle f_j, \phi_j(s) \rangle| &\leq \sum_{k > \eta^{-2}} \sum_{i,j} \sum_{l=0}^{\infty} \left(\sup_{s \in T_{k,i,j,l}} |I(s)|^{-\frac{1}{2}} |\langle f_i, \phi_i(s) \rangle| \right) \\ &\quad \times \prod_{\kappa \neq i} \left(\sum_{s \in T_{k,i,j,l}} |\langle f_{\kappa}, \phi_{\kappa}(s) \rangle|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

Using (40), (41) and Lemma 7 of Section 7 we can bound this by

$$\leq C \sum_{k > \eta^{-2}} 2^{-\sum_j \frac{k}{p_j'}} \sum_{i,j} \sum_{l=0}^{\infty} |J_{T_{k,i,j,l}}|.$$

Now we apply the estimate

$$(42) \quad \sum_{l=0}^{\infty} |J_{T_{k,i,j,l}}| \leq C 2^{10\eta p_j' k} 2^k,$$

for each $k > \eta^{-2}, i, j$, which is proved in Sections 6 and 8. This bounds the previously displayed expression by

$$(43) \quad \leq C \sum_{k > \eta^{-2}} 2^{-\sum \frac{k}{p_j'}} 2^{10\eta p_j' k} 2^k.$$

This is less than a constant since

$$\sum_j \frac{1}{p_j'} \geq 1 + 10\eta \max_j p_j'$$

by the choice of η . This finishes the proof of Proposition 4 up to the proof of estimate (42) and Lemmata 6 and 7.

6. Counting the trees for $i = j$

We prove estimate (42) in the case $i = j$. Thus fix $k > \eta^{-2}, i, j$ with $i = j$. Let \mathcal{F} denote the set of all trees $T_{k,i,j,l}$. Observe that for $T, T' \in \mathcal{F}, T \neq T'$ we have, by (37), that $T \cup T'$ is not a tree; therefore

$$\rho_i(s_T) \cap \rho_i(s_{T'}) = \emptyset.$$

Define $b := 2^{-\eta k} 2^{-\frac{k}{p_i'}}$. Then by (35) for all $T \in \mathcal{F}$

$$(44) \quad |\langle f_i, \phi_i(s_T) \rangle| \geq b |J_T|^{\frac{1}{2}}.$$

Finally recall that for all $s \in S$:

$$(45) \quad I(s) \not\subset \{x : M_{p_i}(Mf_i)(x) \geq \lambda_0\}.$$

Our proof goes in the following four steps:

Step 1. Define the counting function

$$(46) \quad N_{\mathcal{F}}(x) := \sum_{T \in \mathcal{F}} 1_{J_T}(x).$$

We have to estimate the L^1 -norm of the counting function. Since the counting function is integer-valued, it suffices to show a weak-type $1 + \varepsilon$ estimate for small ε . More precisely it suffices to show for all integers $\lambda \geq 1$ and sufficiently small $\delta, \varepsilon > 0$, $\delta = \delta(\eta, p_i), \varepsilon = \varepsilon(\eta, p_i)$:

$$|\{x \in \mathbb{R} : N_{\mathcal{F}}(x) \geq \lambda\}| \leq b^{-p_i' - \delta} \lambda^{-1 - \varepsilon}.$$

Fix such a λ . As in [4] there is a subset $\mathcal{F}' \subset \mathcal{F}$ such that, if we define $N_{\mathcal{F}'}$ analogously to $N_{\mathcal{F}}$,

$$\{x \in \mathbb{R} : N_{\mathcal{F}'}(x) \geq \lambda\} = \{x \in \mathbb{R} : N_{\mathcal{F}}(x) \geq \lambda\}$$

and $\|N_{\mathcal{F}'}\|_{\infty} \leq \lambda$. This is due to the grid structure of $I(S)$.

Step 2. Let $A > 1$ be a number whose value will be specified later. We can write

$$(47) \quad \mathcal{F}' = \left(\bigcup_{m=1}^{A^{10}} \mathcal{F}_m \right) \cup \mathcal{F}''$$

such that if $T, T' \in \mathcal{F}_m$ for some m and $T \neq T'$, then

$$(AJ_T \times \omega(s_T)) \cap (AJ_{T'} \times \omega(s_{T'})) = \emptyset,$$

and

$$(48) \quad \sum_{T \in \mathcal{F}''} |J_T| \leq Ce^{-A} \sum_{T \in \mathcal{F}_1} |J_T|.$$

For a proof of this fact see the proof of the separation lemma in [4].

Step 3. Let $1 \leq m \leq A^{10}$. The following lines hold for all sufficiently small $\delta, \varepsilon > 0$. The arguments may require δ, ε to change from line to line. For a tempered distribution f , $x \in \mathbb{R}$, and $T \in \mathcal{F}_m$ define

$$Bf(x)(T) := \frac{\langle f, \phi_t(s_T) \rangle}{|J_T|^{\frac{1}{2}}} 1_{J_T}(x).$$

Let $L^2(\mathbb{R}, l^2(\mathcal{F}))$ be the Banach space of square-integrable functions on \mathbb{R} with values in $l^2(\mathcal{F})$, and analogously for other exponents. Then we have the following estimate by Lemma 4.3 in [4]

$$\|Bf\|_{L^2(\mathbb{R}, l^2(\mathcal{F}_m))} = \left(\sum_{T \in \mathcal{F}_m} |\langle f, \phi_t(s_T) \rangle|^2 \right)^{\frac{1}{2}} \leq C(1 + A^{-\frac{1}{\varepsilon}} \lambda) \|f\|_2.$$

We also trivially have

$$\begin{aligned} \|Bf\|_{L^{1+\delta}(\mathbb{R}, l^\infty(\mathcal{F}_m))} &= \left(\int \left(\sup_{T \in \mathcal{F}_m: x \in J_T} \frac{|\langle f, \phi_t(s_T) \rangle|}{|J_T|^{\frac{1}{2}}} \right)^{1+\delta} dx \right)^{\frac{1}{1+\delta}} \\ &\leq C \|Mf\|_{1+\delta} \leq C \|f\|_{1+\delta}. \end{aligned}$$

By interpolation we have for $1 < p < 2$:

$$\|Bf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_m))} \leq C(1 + A^{-\frac{1}{\varepsilon}} \lambda) \|f\|_p.$$

Let $J \in I(S)$, and let $\mathcal{F}_{m,J}$ be the set of $T \in \mathcal{F}_m$ such that $J_T \subset J$. By a localization argument, as in [4], we see that

$$\|Bf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_{m,J}))} \leq C\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}} \lambda) |J|^{\frac{1}{p}} \inf_{x \in J} M_p(Mf)(x).$$

In the following, g^\sharp denotes the sharp maximal function of g with respect to the given grid, as in [4]. We define $N_{\mathcal{F}_m, J}$ in analogy to (46) to be the counting function of the trees $T \in \mathcal{F}_m$ for which $I_T \subset J$. We apply the previous estimate for f_i and use (44) to obtain

$$\begin{aligned} \left(N_{\mathcal{F}_m}^{\frac{p}{p'+\delta}}\right)^\sharp(x) &\leq \sup_{J:x \in J} \left(\frac{1}{|J|} \int_J N_{\mathcal{F}_m, J}(x)^{\frac{p}{p'+\delta}} dx \right) \\ &\leq b^{-p} \sup_{J:x \in J} \frac{1}{|J|} \left\| \left(\sum_{T \in \mathcal{F}_m, J} \frac{|\langle f_i, \phi_i(s_T) \rangle|^{p'+\delta}}{|J_T|^{\frac{p'+\delta}{2}}} 1_{J_T} \right)^{\frac{1}{p'+\delta}} \right\|_p^p \\ &\leq b^{-p} C \left(\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}} \lambda) M_p(Mf_i)(x) \right)^p. \end{aligned}$$

Using (45) we can sharpen this argument in the case $p = p_i$ to

$$\left(N_{\mathcal{F}_m}^{\frac{p_i}{p_i'+\delta}}\right)^\sharp(x) \leq C b^{-p_i} \left(\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}} \lambda) \min\{M_{p_i}(Mf_i)(x), \lambda_0\} \right)^{p_i}.$$

Taking the $\frac{p_i'+2\delta}{p_i}$ -norm on both sides and raising to the $\frac{p_i'+\delta}{p_i}$ -th power gives

$$(49) \quad \|N_{\mathcal{F}_m}\|_{\frac{p_i'+2\delta}{p_i}} \leq C b^{-p_i'-\delta} \left(\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}} \lambda) \right)^{p_i'+\delta}.$$

Step 4. We split the counting function $N_{\mathcal{F}'}$ according to (47) and use the weak-type estimate following from (49) on the first part and estimate (48) together with (49) and the fact that the counting function is integer-valued on the second part. This gives

$$\begin{aligned} \{x \in \mathbb{R} : N_{\mathcal{F}'}(x) \geq A^{10} \lambda\} &\leq C A^{10} \lambda^{-\frac{p_i'+2\delta}{p_i'+\delta}} b^{-p_i'-2\delta} \left(\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}} \lambda) \right)^{p_i'+2\delta} \\ &\quad + e^{-A} C b^{-p_i'-2\delta} \left(\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}} \lambda) \right)^{p_i'+2\delta}. \end{aligned}$$

Choosing A of the order λ^ε and $\varepsilon \ll \delta$ gives

$$\{x \in \mathbb{R} : N_{\mathcal{F}'}(x) \geq \lambda\} \leq C \lambda^{-1-\varepsilon} b^{-p_i'-\delta}.$$

According to Step 1 this finishes the proof of estimate (42) in the case $i = j$.

7. Estimates on a single tree

This section collects some standard facts from Calderón-Zygmund theory, adapted to the setup of trees.

LEMMA 6. *Fix k, i, j, l such that $T := T_{k,i,j,l}$ is a tree, assume $i \neq j$, and let $1 < p \leq 2$. We then have*

$$(50) \quad \left\| \left(\sum_{s \in T} \frac{|\langle f, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_p \leq C \|f\|_p.$$

For each interval $J \in I(S)$ define $T_J := \{s \in T : I(s) \subset J\}$. Then we obtain

$$(51) \quad \left\| \left(\sum_{s \in T_J} \frac{|\langle f, \phi_J(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_p \leq C |J|^{\frac{1}{p}} \inf_{x \in J} M_p(Mf)(x).$$

For each $s \in T$, let h_s be a measurable function supported in $I(s)$ with $\|h_I(x)\|_\infty = |I(s)|^{-\frac{1}{2}}$, $\|h\|_2 = 1$, and $\langle h_s, h_{s'} \rangle = 0$ for $s \neq s'$. Then for all maps $\varepsilon : T \rightarrow \{-1, 1\}$, we have

$$(52) \quad \left\| \sum_{s \in T} \varepsilon(s) \langle f, \phi_J(s) \rangle h_s \right\|_p \leq C \|f\|_p.$$

First we prove estimate (52). The estimate is true in the case $p = 2$, as is proved in [4]. By interpolation it suffices to prove the weak-type estimate

$$(53) \quad \left| \left\{ x \in \mathbb{R} : \sum_{s \in T} \varepsilon(s) \langle f, \phi_J(s) \rangle h_s(x) \geq C\lambda \right\} \right| \leq C' \frac{\|f\|_1}{\lambda}.$$

Let $f \in L^1(\mathbb{R})$. We write f as the sum of a good function g and a bad function b as follows. Let $\{I_n\}_n$ be the set of maximal intervals of the grid $I(S)$ for which

$$\int_{I_n} |f(x)| dx \geq \lambda |I_n|.$$

Let $\xi \in \omega_i(s_T)$, and pick a function $\theta_{\xi, i}$ as in hypothesis (iv) of Proposition 3. For each of the intervals I_n , define

$$b_n(x) := 1_{I_n}(x) (f(x) - \lambda_n \theta_{\xi, i}(x)),$$

where λ_n is chosen such that b_n is orthogonal to $\theta_{\xi, i}$. Obviously λ_n is bounded by $C\|f(x)\|_{L^1(I_n)}$. Define $b := \sum_n b_n$ and $g := f - b$. It suffices to prove estimate (53) for the good and bad function separately. The estimate for the good function follows immediately from estimate (52) for $p = 2$. For the bad function we proceed as follows. Since the set

$$E := \bigcup_n 2I_n$$

is bounded in measure by $C\lambda^{-1}$, it suffices to prove the strong-type estimate

$$(54) \quad \left\| \sum_n \left(\sum_{s \in T} \varepsilon(s) \langle b_n, \phi_J(s) \rangle h_s \right) \right\|_{L^1(E^c)} \leq C \|f\|_1.$$

We estimate each summand separately. Obviously, we have

$$\left\| \sum_{s \in T} \varepsilon(s) \langle b_n, \phi_J(s) \rangle h_s \right\|_{L^1(E^c)} \leq \sum_{s \in T : I(s) \not\subset 2I_n} |I(s)|^{\frac{1}{2}} |\langle b_n, \phi_J(s) \rangle|.$$

For each integer k let T_k be the set of those $s \in T$, for which $|I(s)| \leq 2^k |I_n| < 2|I(s)|$ and $I(s) \not\subset 2I_n$. For $k < 2$ we use the estimate

$$\begin{aligned}
 (55) \quad & \sum_{s \in T_k} |I(s)|^{\frac{1}{2}} |\langle b_n, \phi_j(s) \rangle| \\
 & \leq C \|b_n\|_1 \sum_{s \in T_k} \left(1 + \frac{|c(I(s)) - c(I_n)|}{|I(s)|} \right)^{-2} \\
 & \leq C \|b_n\|_1 \int_{(2I_n)^c} \sum_{s \in T_k} \frac{1}{2^k |I_n|} \left(1 + \frac{|x - c(I_n)|}{2^k |I_n|} \right)^{-2} 1_{I(s)}(x) dx \\
 & \leq C \|b_n\|_1 2^k.
 \end{aligned}$$

For the last inequality we have seen that the intervals $I(s)$ with $s \in T_k$ are pairwise disjoint.

For $k > 2$ we use the orthogonality of b_n and $\theta_{\xi, i}$ as well as hypothesis (iv) of Proposition 3 to obtain

$$\begin{aligned}
 (56) \quad & \sum_{s \in T_k} |I(s)|^{\frac{1}{2}} |\langle b_n, \phi_j(s) \rangle| \leq \sum_{s \in T_k} |I(s)|^{\frac{1}{2}} \|b_n\|_1 \inf_{\lambda} \|\phi_j(s) - \lambda \theta_{\xi, i}\|_{L^\infty(I_n)} \\
 & \leq C \|b_n\|_1 \sum_{s \in T_k} \left(1 + \frac{|c(I(s)) - c(I_n)|}{|I(s)|} \right)^{-2} \frac{|I_n|}{|I(s)|} \\
 & \leq C \|b_n\|_1 2^{-k}.
 \end{aligned}$$

The last inequality follows by a similar argument as in the case $k \leq 2$. Summing (55) and (56) over k and n gives (54) and finishes the proof of (52).

We prove estimate (50). Observe that (52) is not void, since functions h_s clearly exist. Therefore we can average (52) over all choices of ε to obtain:

$$\begin{aligned}
 2^{-|T|} \sum_{\varepsilon} \left\| \sum_{s \in T} \varepsilon(s) \langle f, \phi_j(s) \rangle h_s \right\|_p^p &= \int_{\mathbb{R}} 2^{-n} \sum_{\varepsilon} \left(\sum_{s \in T} \varepsilon(s) \langle f, \phi_j(s) \rangle h_s(x) \right)^p dx \\
 &\leq C \|f\|_p^p.
 \end{aligned}$$

Now Khinchine's inequality gives

$$\int_{\mathbb{R}} \left(2^{-n} \sum_{\varepsilon} \left(\sum_{s \in T} \varepsilon(s) \langle f, \phi_j(s) \rangle h_s(x) \right)^2 \right)^{\frac{p}{2}} dx \leq C \|f\|_p^p,$$

which immediately implies estimate (50).

To prove (51) fix a J and write $f = f 1_{2J} + f 1_{(2J)^c}$. It suffices to prove the estimate separately for both summands. For the first summand we simply

apply (50). For the second summand we write

$$\begin{aligned} \left(\sum_{s \in T_J} \frac{|\langle f 1_{(2J)^c}, \phi_J(s) \rangle|^2}{|I(s)|} 1_{I(s)}(x) \right)^{\frac{1}{2}} &\leq C \sum_{s \in T_J: x \in I(s)} Mf(x) |I(s)| |J|^{-1} \\ &\leq CMf(x) 1_J(x). \end{aligned}$$

The last inequality follows by summing a geometric series. This proves the estimate for the second summand and finishes the proof of Lemma 6.

LEMMA 7. *Fix $k \geq \eta^{-2}, \iota, j, l$ such that $T := T_{k, \iota, j, l}$ is a tree and assume $\iota \neq j$. Then we have*

$$(57) \quad \left(\sum_{s \in T} |\langle f_j, \phi_j(s) \rangle|^2 \right)^{\frac{1}{2}} \leq C \left\| \left(\sum_{s \in T} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 |J_T|^{-\frac{1}{2}}.$$

Proof. For each $J \in I(S)$,

$$(58) \quad \frac{1}{|J|} \int_J \left(\sum_{s \in T: I(s) \subset J} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)}(x) \right)^{\frac{1}{2}} dx \leq C 2^{-\frac{k}{p_j'}},$$

since the set $\{s \in T : I(s) \subset J\}$ is a union of trees $\{T_n\}_n$ which satisfy (41) for $k-1$ and

$$\sum_n |J_{T_n}| \leq |J_T|.$$

Define for $x \in \mathbb{R}$ and $s \in T$:

$$F(x)(s) := \sum_{s \in T} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)}(x).$$

Since F is supported on J_T , we have

$$\|F\|_{L^2(\mathbb{R}, l^2(T))} \leq |J_T|^{\frac{1}{2}} \|F\|_{\text{BMO}(\mathbb{R}, l^2(T))}.$$

Here BMO is understood with respect to the grid $I(S)$ as in [4]. We prove Lemma 7 by estimating this BMO-norm with (58) and (36).

8. Counting the trees for $\iota \neq j$

We prove estimate (42) in the case $\iota \neq j$. Thus fix $k \geq \eta^{-2}, \iota, j$ with $\iota \neq j$. Let \mathcal{F} denote the set of all trees $T_{k, \iota, j, l}$.

As in [4] we define for $T \in \mathcal{F}$:

$$\begin{aligned} T^{\min} &:= \{s \in T : \rho_i(s) \text{ is minimal in } \rho_i(T)\}, \\ T^{\text{fat}} &:= \{s \in T : 2^5 2^{\eta k} |I(s)| \geq |J_T|\}, \\ T^\partial &:= \{s \in T : I(s) \cap (1 - 2^{-4})J_T = \emptyset\}, \\ T^{\partial \max} &:= \{s \in T^\partial : \rho_i(s) \text{ is maximal in } \rho_i(T^\partial)\}, \\ T^{\text{nice}} &:= T \setminus (T^{\min} \cup T^{\text{fat}} \cup T^\partial). \end{aligned}$$

Define $b := 2^{-\frac{k}{p_j'}}$. By similar arguments as in [4] we have the estimate

$$(59) \quad \text{if } i \neq j, \text{ then } \left\| \left(\sum_{s \in T^{\text{nice}}} \frac{|\langle f_j, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)} \right)^{\frac{1}{2}} \right\|_1 \geq b |J_T|.$$

Define the counting function

$$N_{\mathcal{F}}(x) := \sum_{T \in \mathcal{F}} 1_{J_T}(x).$$

As in Section 6 it suffices to show

$$(60) \quad |\{x \in \mathbb{R} : N_{\mathcal{F}}(x) \geq \lambda\}| \leq b^{-p_j' - \delta} \lambda^{-1 - \varepsilon}$$

for all integers $\lambda \geq 1$ and small $\varepsilon, \delta > 0$. In addition, we can assume that $\|N_{\mathcal{F}}\|_\infty \leq \lambda$.

Let $y \in \mathbb{R}$, $T \in \mathcal{F}$, $x \in J_T$, and $s \in T$. For $f \in \mathcal{S}(\mathbb{R})$ define

$$Sf(y)(T)(x)(s) := \frac{\langle f, \phi_j(s) \rangle}{|I(s)|^{\frac{1}{2}}} 1_{I(s)}(x) 1_{J_T}(y).$$

Consider J_T as a measure space with Lebesgue measure normalized to 1. Then the operator is bounded from L^2 to $L^2(\mathbb{R}, l^2(\mathcal{F}, (L^2(J_T, l^2(T)))))$, as we see below. We have used a sloppy notation for the second Banach space: The range space $L^2(J_T, l^2(T))$ depends on the variable $T \in \mathcal{F}$. To make this space independent of T , we take the direct sum of these Banach spaces as T varies over \mathcal{F} , and we let $Sf(y)(T)$ be nonzero only on the component corresponding to T . This is how we interpret the above notation. To see the claimed estimate we calculate:

$$\begin{aligned} \int \sum_{T \in \mathcal{F}} \frac{1}{|J_T|} \int \sum_{s \in T} \frac{|\langle f, \phi_j(s) \rangle|^2}{|I(s)|} 1_{I(s)}(x) 1_{J_T}(y) dx dy &= \sum_{s \in \bigcup_{T \in \mathcal{F}} T} |\langle f, \phi_j(s) \rangle|^2 \\ &\leq C(1 + \lambda A^{-\frac{1}{\varepsilon}}) \|f\|_2^2, \end{aligned}$$

the last inequality being taken from [4]. The operator is also bounded from $L^{1+2\delta}$ into

$$L^{1+2\delta}(\mathbb{R}, l^\infty(\mathcal{F}, L^{1+\delta}(J_T, l^2(T))))$$

since by Lemma 6:

$$\begin{aligned}
& \int \left(\sup_{T \in \mathcal{F}} \left(\frac{1}{|J_T|} \int \left(\sum_{s \in T} \left(\frac{|\langle f, \phi_j(s) \rangle|}{|I(s)|^{\frac{1}{2}}} 1_{I(s)}(x) 1_{J_T}(y) \right)^2 \right)^{\frac{1+\delta}{2}} dx \right)^{\frac{1}{1+\delta}} \right)^{1+2\delta} dy \\
& \leq \int \left(\sup_{T \in \mathcal{F}: y \in J_T} \frac{1}{|J_T|} \left\| \left(\sum_{s \in T} \left(\frac{|\langle f, \phi_j(s) \rangle|}{|I(s)|^{\frac{1}{2}}} 1_{I(s)} \right)^2 \right)^{\frac{1}{2}} \right\|_{1+\delta}^{1+\delta} \right)^{\frac{1+2\delta}{1+\delta}} dy \\
& \leq C \int (M_{1+\delta}(Mf)(y))^{1+2\delta} dy \\
& \leq C \|f\|_{1+2\delta}^{1+2\delta}.
\end{aligned}$$

By complex interpolation and the fact that $L^q(J_T) \subset L^1(J_T)$ for $q \geq 1$ we obtain that S maps L^p into $L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}, L^1(J_T, l^2(T))))$ with norm less than $C(1 + \lambda A^{-\frac{1}{\varepsilon}})$.

Let $J \in I(S)$ and define \mathcal{F}_J to be the set of $T \in \mathcal{F}$ such that $J_T \subset J$. Then we can localize as before to get

$$\|Sf\|_{L^p(\mathbb{R}, l^{p'+\delta}(\mathcal{F}_J, L^1(J_T, l^2(T))))} \leq C\lambda^\varepsilon (1 + \lambda A^{-\frac{1}{\varepsilon}}) |J|^{\frac{1}{p}} \inf_{x \in J} M^p(Mf)(x).$$

Using the estimate (59) on nice trees gives, for $f = f_j$ and $p = p_j$,

$$\begin{aligned}
\left(N_{\mathcal{F}}^{\frac{p_j}{p_j'+\varepsilon}} \right)^\#(x) & \leq \sup_{J: x \in J} \left(\frac{1}{|J|} \int_J N_{\mathcal{F}_J}(x)^{\frac{p_j}{p_j'+\varepsilon}} dx \right) \\
& \leq b^{-p_j} \sup_{J: x \in J} \left(\frac{1}{|J|} \|Sf_j\|_{L^{p_j}(\mathbb{R}, l^{p_j'+\delta}(\mathcal{F}_J, L^1(J_T, l^2(T))))}^{p_j} \right) \\
& \leq b^{-p_j} C\lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}}\lambda)^{p_j} (M_{p_j}(Mf_j)(x))^{p_j}.
\end{aligned}$$

Again we can sharpen this argument to obtain

$$\left(N_{\mathcal{F}}^{\frac{p_j}{p_j'+\varepsilon}} \right)^\#(x) \leq C b^{-p_j} \lambda^\varepsilon (1 + A^{-\frac{1}{\varepsilon}}\lambda)^{p_j} \max\{M_{p_j}(Mf_j)(x)^{p_j}, \lambda_0\}.$$

Taking the $\frac{p_j'+\delta}{p_j}$ -norm on both sides proves estimate (60) and therefore also (42).

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(Received May 20, 1997)